Connected Perfect Domination in Bipolar fuzzy graphs

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Abstract: In this paper we introduced and studied the concepts of connected domination $\gamma_c(G)$ and tribal connected domination $\gamma_{tc}(G)$ in bipolar fuzzy graphs. We determine the connected perfect domination number for some standard fuzzy graphs with suitable examples. We investigated the relationship of $\gamma_c(G)$ and $\gamma_{tc}(G)$ with other known parameters of G. Some bounds and some interesting results for the parameters are obtained. Further, we also obtain Nordhaus - Gaddum type results of $\gamma_c(G)$. finally, the tribal connected perfect domination number$\gamma_{tcp}(G)$ for several classes of biploar fuzzy graphs are given with suitable example and obtain bounds for the same.
Keywords: Intuitionistic Fuzzy graph,bipolar fuzzy graph. domination number, perfect domination, connected perfect domination and tribal connected domination number.
2010 AMS Mathematics Subject Classification: 05C72, 03F55, 05C69

1 Introduction

Graph theory is an important part of mathematics. In the biginning of nineteenth century Euler first introduced the concept of theory graph [8]. In the history of mathematics the solution given by Euler of the Well-known Konigsberg bridge problems is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science. (1965) Zadeh [22] was introduced the notion of a fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences engineering, statistic, graph theory, artificial intelligence, signal processing, multiagent systems, decision making and automata theory. (1975) Rosenfeld [16] discussed the concept of fuzzy graphs while the basic idea of fuzzy graph was introduced by Kauffmann in (1973) [9]. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. (1994) Zhang [24] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. A bipolar fuzzy set is an extension of Zadeh idea of fuzzy set theory whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $[0, 1]$ of an

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element indicates that the element some what satisfies the property, and the membership degree $[-1,0]$ of an element indicates that the element somewhat satisfies the implicit counter-property. The notion of bipolar fuzzy graphs was introduced by Akram [1] and further was studied in [3]. Again Akram investigated regular bipolar fuzzy graphs [3]. Forother related works of bipolar sets see [2, 4, 21, 23]. (2016) [5] M. Akram and A. Farooq introduced the concept of bipolar fuzzy Trees. (2019) [6] B. A. Mohideen introduced Perfect bipolar fuzzy graphs.
The concept of domination in bipolar fuzzy graphs was investigated by M. G. Karunambigai, M. Akram and K. Palanivel in (2013) [10] and investigated the concept domination independence and irredundance on bipolar fuzzy graph. In (2016) [12] V. Mohanaselvi, S. Sivamani introduced the concept of domination in bipolar fuzzy graphs. (2017) [20] D. Umamageswari, P. Thangaraj, investigated the concept of domination in operation on bipolar fuzzy graphs. At the same year R. Muthuraj and Kanimozhi investigated the concept of Total strong (weak) domination in bipolar fuzzy graph [14].
The problem of selecting two disjoint sets of transmitting stations so that one set can provide service in the case of failure of some of the transmitting stations of the other set. This led them to define the inverse domination number. In this aspect, it is worthwhile to concentrate on dominating and inverse dominating sets.
The perfect domination in bipolar fuzzy graphs was introduced by R. Muthuraj in (2018) [13]. In this paper we introduced and investigate the concepts of connected perfect dominating and trouble connected domination in bipolar fuzzy graphs. And obtain many result related to this concepts and relationship between this concepts and the others in bipolar fuzzy graph will be given with suitable example.

2 Basic Definitions

In this section, contains some basic definitions and theorems relating to fuzzy graphs, bipolar fuzzy graphs, domination and perfect domination are given.
A crisp graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. A fuzzy graph $G=(\mu,\rho)$ is a set with two function $\mu: V \to [0,1]$ and $\rho: E \to [0,1]$ such that $\rho(\{x,y\}) \leq \mu(x) \wedge \mu(y)$ for all $x,y \in V$. We write $\rho(x,y)$ for $\rho(\{x,y\})$. The order p and size q of a fuzzy graph $G=(\mu,\rho)$ are defined to be $p=\sum_{x\in V}\mu(x)$ and $q=\sum_{xy\in E}\rho(x,y)$.
By a bipolar fuzzy graph, we mean a pair of the form $G=(A,B)$, where $A=(\mu_A^+,\mu_A^-)$ is a bipolar fuzzy set in $V=\{v_1,v_2,.....,v_n\}$ and $B=(\rho_B^+,\rho_B^-)$ is a bipolar relation on V such that.

$$\rho_B^p = \rho_B^p(u,v) \leq \mu_A^p(u) \wedge \mu_A^p(v),$$

and

$$\rho_B^N = \rho_B^N(u,v) \geq \mu_A^N(u) \vee \mu_A^N(v).$$

for all $(v,u) \in E$.
We call A the bipolar fuzzy vertex set of V, B the bipolar fuzzy edge set of E, respectively. Note that B is symmetric bipolar fuzzy relation on A. We use the notation uv for an element of E.Thus, $G=(A,B)$ is a bipolar graph of $G^*=(V,E)$ if

$$\rho_B^p = \rho_B^p(u,v) \leq \mu_A^p(u) \wedge \mu_A^p(v),$$

and

$$\rho_B^N = \rho_B^N(u,v) \geq \mu_A^N(u) \vee \mu_A^N(v).$$

.
for all $(v,u) \in E$. In a bipolar fuzzy graph G, when $\mu_{2ij}^{p} = \mu_{2ij}^{N} = 0$ for some i and j, then there is no edge between v_i and v_j, otherwise there exists an edge between v_i and v_j.
A bipolar fuzzy graph, $BFG\ G = (A, B)$ is said to be a $semi - \mu_2^P$ strong bipolar fuzzy graph if $\mu_{2ij}^{p} = min\{\mu_{1i}^{p}, \mu_{1j}^{p}\}$ for every i and j. A bipolar fuzzy graph $BFG\ G = (V, E)$ is said to be a semi-μ_2^N strong bipolar fuzzy graph if $\mu_{2ij}^{N} = max\{\mu_{1i}^{N}, \mu_{1j}^{N}\}$ for every i and j.
Let $G = (A, B)$ be a bipolar fuzzy graph Then, the cardinality of G is defined to be

$$|G| = |\sum_{v_i \in V} \frac{1 + \mu_A^P(v_i) + \mu_A^N(v_i)}{2} + \sum_{(v_i,v_j) \in E} \frac{1 + \rho_B^P(v_i, v_j) + \rho_B^N(v_i, v_j)}{2}|$$

.
Let $G = (A, B)$ be a bipolar fuzzy graph. Then the vertex cardinality of G is defined by

$$|V| = \sum_{v_i \in V} \frac{1 + \mu_A^P(v_i) + \mu_A^N(v_i)}{2}$$

for all $v_i \in V$. is called the order of a bipolar fuzzy graph $G = (A, B)$ and is denoted by $p(G)$ or $O(G) = (o^N(G), o^p(G))$, $o^P(G) = \sum_{u \in V} \mu_A^P(u)$, $o^N(G) = \sum_{u \in V} \mu_A^N(u)$. The edge cardinality of a bipolar fuzzy graph G is defined by

$$|E| = \sum_{(v_i,v_j) \in E} \frac{1 + \rho_B^P(v_i, v_j) + \rho_B^N(v_i, v_j)}{2}.$$

for all $(v_i, v_j) \in E$ is called the size of a bipolar fuzzy graph, $G = (A, B)$ and is denoted by $q(G)$, or $S(G) = (S^N(G), S^P(G))$, $S^P(G) = \sum_{uv \in E} \rho_B^P(uv)$, $S^N(G) = \sum_{uv \in E} \rho_B^N(uv)$. An edge $e = (x, y)$ of a bipolar fuzzy graph is called an effective edge if,

$$\mu_2^P(x, y) = min\{\mu_1^P(x), \mu_1^P(y)\},$$

and

$$\mu_2^N(x, y) = max\{\mu_1^N(x), \mu_1^N(y)\}.$$

The degree of a vertex can be generalized in different ways for a bipolar fuzzy graph $G = (V, E)$. The effective degree of a vertex v in a bipolar fuzzy graph, $G = (V, E)$ is defined to be sum of the weights of the effective edges incident at v and it is denoted by $d_E(v)$. The minimum effective degree of G is $\delta_E(G) = min\{d_E(v)|v \in V\}$. The maximum effective degree of G is $\Delta_E(G) = max\{d_E(v)|v \in V\}$.
Two vertices v_i and v_j are said to be neighbors in a bipolar fuzzy graph $G = (V, E)$ if either one of the following conditions holds, (1) $\mu_2^P(v_i, v_j) > 0$ and $\mu_2^N(v_i, v_j) < 0$.
(2) $\mu_2^P(v_i, v_j) = 0$ and $\mu_2^N(v_i, v_j) < 0$.
(3) $\mu_2^P(v_i, v_j) > 0$ and $\mu_2^N(v_i, v_j) = 0; v_i, v_j \in V$.
A vertex subset $N(v) = \{u \in V : v$ adjacent to $u\}$ is called the open neighborhood set of a vertex v and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood set of v.
The neighborhood degree of a vertex v in a bipolar fuzzy graph, $G = (V, E)$ is defined to be sum of the weights of the vertices adjacent to v, and it is denoted by $d_N(v)$, that is mean that $d_N(v) = |N(v)|$.
The minimum neighborhood degree of G is $\delta_N(G) = min\{d_N(v)|v \in V\}$.
The maximum neighborhood degree of G is $\Delta_N(G) = max\{d_N(v)|v \in V\}$.
A bipolar fuzzy graph, $G = (V, E)$ is said to be complete bipolar fuzzy graph if

$\mu_2^P(v_i,v_j) = min\{\mu_1^P(v_i), \mu_1^P(v_j)\}$, $\mu_2^N(v_i,v_j) = max\{\mu_1^N(v_i), \mu_1^N(v_j)\}$.
for all $v_i, v_j \in V$ and is denoted by K_p.
The complement of a bipolar fuzzy graph, $G = (V, E)$ is a bipolar fuzzy graph $\overline{G} = (\overline{V}, \overline{E})$, where
(i) $\overline{V} = V$;
(ii) $\overline{\mu_{1i}^P} = \mu_{1i}^P$; $\overline{\mu_{1i}^N} = \mu_{1i}^N$ for all $i = 1, 2, 3, ..., n$.
(iii) $\overline{\mu_{2ij}^P} = min\{\mu_{1i}^P, \mu_{1j}^P\} - \mu_{2ij}^P$ and $\overline{\mu_{2ij}^N} = max\{\mu_{1i}^N, \mu_{1j}^N\} - \mu_{2ij}^N$ for all $i, j = 1, 2, 3, ..., n$.
A bipolar fuzzy graph $G = (V, E)$ is said to bipartite if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that
(i) $\mu_2^P(v_i, v_j) = 0$ and $\mu_2^N(v_i, v_j) = 0$ if $v_i, v_j \in V_1$ or $v_i, v_j \in V_2$;
(ii) $\mu_2^P(v_i, v_j) > 0$, and $\mu_2^N(v_i, v_j) < 0$, if $v_i \in V_1$ and $v_j \in V_2$, for some i and j, (or);
$\mu_2^P(v_i, v_j) = 0$ and $\mu_2^N(v_i, v_j) < 0$, if $v_i \in V_1$ and $v_j \in V_2$, (or);
$\mu_2^P(v_i, v_j) > 0, \mu_2^N(v_i, v_j) = 0$ if $v_i \in V_1$ and $v_j \in V_2$. for some i and j.
A bipartite bipolar fuzzy graph, $G = (A, B)$ is said to be complete bipartite bipolar fuzzy graph if $\mu_2^P(v_i, v_j) = min\{\mu_1^P(v_i), \mu_1^P(v_j)\}$ and $\mu_2^N(v_i, v_j) = max\{\mu_1^N(v_i), \mu_1^N(v_j)\}$ for all $v_i \in V_1$, and $v_j \in V_2$. Its denoted by $K_{m,n}$, where $|V_1| = m$, $|V_2| = n$.
A vertex $u \in V$ of a bipolar fuzzy graph, $G = (A, B)$ is said to be an isolated vertex if $\mu_2^P(v, u) = 0$ and $\mu_2^N(v, u) = 0$ for all $v \in V$. That is $N(u) = \phi$. Thus, an isolated vertex does not dominate any other vertex in G.
A subset D of V in a fuzzy graph $G = (\mu, \rho)$ is called a dominating set in G if for every $v \in V - D$, there exists $u \in D$ such that u dominates v. The minimum fuzzy cardinality of dominating sets in G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of fuzzy cardinality $\gamma(G)$ is called a minimum dominating set or $\gamma - set$. A dominating set D of a fuzzy graph G is said to be a minimal dominating set if no proper subset of S is a dominating set of G. The maximum fuzzy cardinality of minimal dominating sets is called the upper domination number of G and is denoted by $\Gamma(G)$. Let $G = (\mu, \rho)$ be a fuzzy graph on V. A subset D of V is said to be an independent set if $\rho(u, v) < \mu(u) \wedge \mu(v)$ for all $u, v \in D$. The maximum fuzzy cardinality of independent sets in G is called the independence number of G and is denoted by $\beta(G)$. A dominating set D of a fuzzy graph (μ, ρ) is called connected dominating set of G if the induced fuzzy subgraph $< D >$ is connected. The connected domination number of a fuzzy graph G is the minimum fuzzy cardinality of connected dominating sets of G and is denoted by $\gamma_c(G)$. A dominating set D of G is called independent dominating set if D is independent. The independence domination number of G is the minimum fuzzy cardinality of the independent dominating sets and is denoted by $\gamma_i(G)$.
A dominating set S of a a bipolar fuzzy graph is said to be minimal dominating set if for each vertex $v \in S$, $V - \{v\}$ is not a dominating set.
The minimum fuzzy cardinality among all minimal dominating set is called lower domination number of G, and is denoted by $\gamma(G)$.
The maximum fuzzy cardinality among all minimal dominating set is called upper domination number of G, and is denoted by $\Gamma(G)$.
This concept was introduced and studied by Kulli and Domke in[?] and [?], respectively.
Let $G = (\mu, \rho)$ be a fuzzy graph and let $u, v \in V$. We say u dominates v in G if $\rho(u, v) = \mu(u) \wedge \mu(v)$.
A subset D of V is called a dominating set in G if for every $v \in V - D$, there exists $u \in D$ such that u dominates v.
The minimum fuzzy cardinality of dominating sets in G is called the domination number of G and is denoted by $\gamma(G)$ or simply γ.
A dominating set D of a fuzzy graph $G = (\mu, \rho)$ is a connected dominating set if the

induced fuzzy subgraph $H = (< D >, \mu', \rho')$, is connected.
The minimum fuzzy cardinality of connected dominating sets of G is called the connected domination number of G and is denoted by $\gamma_c(G)$, or simply γ_c. Let G be a fuzzy graph without isolated vertex, then a subset D of V is said to be a total dominating set of G if every vertex in V is dominated by a vertex in D.
The minimum fuzzy cardinality of total dominating sets of G is called the total domination number of G and is denoted by $\gamma_t(G)$. These concepts were introduced and studied by A. Somasundaran and S. Somasundaram in [19, 18].
A bipolar fuzzy graph $G = (A, B)$ is called strong bipolar fuzzy graph, if

$$\mu_{2ij}^p = \mu_{2ij}^p(v_i, v_j) = \mu_1^p(v_i) \wedge \mu_1^p(v_j),$$

and

$$\mu_{2ij}^N = \mu_{2ij}^N(v_i, v_j) = \mu_1^N(v_i) \vee \mu_1^N(v_j).$$

for all $(v_i, v_j) \in E(G)$.
Let $G = (A, B)$ be a bipolar fuzzy graph and $u, v \in V(G)$, we say that u dominates v in G

$$\mu_{2ij}^p = \mu_2^p(u, v) = \mu_1^p(u) \wedge \mu_1^p(v),$$

and

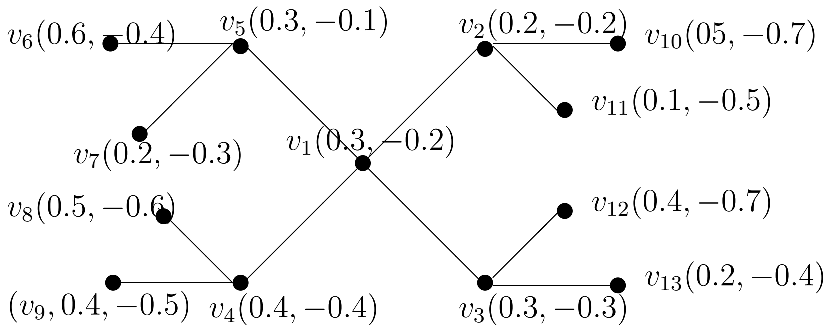
$$\mu_{2ij}^N = \mu_2^N(u, v) = \mu_1^N(u) \vee \mu_1^N(v).$$

for all $(v_i, v_j) \in E(G)$.
A vertex subset D of V in bipolar fuzzy graph G is called a dominating set in G if for every $v \in V - S$, there exists $u \in D$ such that u dominates v.
Let $G = (A, B)$ be a bipolar fuzzy graph. Let $x, y \in V$. The vertex x dominates the vertex y in G if (x, y) is a strong arc or strong edge. A subset D of A is called a perfect dominating set of G if for each vertex y is not in D is dominated by exactly one vertex of D.
A perfect dominating set D of a bipolar fuzzy graph G is said to be a minimal perfect dominating set, if for each vertex y in D, $D - \{y\}$ is not a perfect dominating set of a bipolar fuzzy graph G.
The minimum fuzzy cardinality of a minimal perfect dominating set of a bipolar fuzzy graph G is called the perfect domination number of a bipolar fuzzy graph G. It is denoted by $\gamma_p(G)$.
The maximum fuzzy cardinality of a minimal perfect dominating set of a bipolar fuzzy graph G is called the upper perfect domination number of a bipolar fuzzy graph G. It is denoted by $\Gamma_p(G)$.

3 Connected Perfect Domination in bipolar Fuzzy Graphs

Definition 3.1: The perfect dominating set D of a bipolar fuzzy graph$G = (A, B)$ is connected perfect dominating set of G if $< D >$ the bipolar fuzzy subgraph induced by D is connected.
Definition 3.2: The connected perfect dominating set in a bipolar fuzzy graph $G = (A, B)$ is called minimal connected perfect dominating set of G if for every $u \in D$, $D - \{u\}$ is not connected perfect dominating set of G.
Definition 3.3: The minimum fuzzy cardinality of a minimal connected perfect dominating set of a bipolar fuzzy graph G is called the connected perfect domination number

of a bipolar fuzzy graph G. It is denoted by $\gamma_{cpb}(G)$.
The maximum fuzzy cardinality of a minimal connected perfect dominating set of a bipolar fuzzy graph G is called the upper perfect domination number of a bipolar fuzzy graph G. It is denoted by $\Gamma_{cpb}(G)$.
Example 3.1: Consider a strong bipolar fuzzy graph $G = (A, B)$ given in Figure 3.1.



$Fig.$ 3.1

We verify that the vertex subsets $D = \{v_1, v_2, v_3, v_4, v_5\}$ is minimal connected perfect dominating set of G and hence, $\gamma_c p(G) = \{|D|\} = 2.65$.
Here the upper connected perfect domination number equal to $\Gamma_c p(G) = \gamma_c p(G) = \{|D|\} = 2.65$.
Theorem 3.1: Every connected perfect dominating set in a bipolar fuzzy graph $G = (A, B)$ is a perfect dominating set of a bipolar fuzzy graph G.
Proof: We know that a dominating set D is connected perfect dominating set in a bipolar fuzzy graph G. if for each vertex v not in D and v dominated by exactly one vertex of D and also the induced bipolar fuzzy subgraph $< D >$ is connected it is clear that every vertex v not in D and v is dominated by exactly one vertex of D which is a perfect dominating set of a bipolar fuzzy graph G. Therefore every connected perfect dominating set of a bipolar fuzzy graph G is a perfect dominating set of G.
Theorem 3.2: Let $G = (A, B)$ be a connected bipolar fuzzy graph. Let D be a minimal connected perfect dominating set of G. Then $V - D$ is not a connected perfect dominating set of G.
Proof: Let D be a minimal connected perfect dominating set in bipolar fuzzy graph G. Let v be any vertex of D. Since G is connected. Then by theorem 3.1 [], G has no isolated vertices, there exists a vertex $u \in N(v)$. u must be dominated by at least one vertex in $D - \{v\}$, (i.e) $D - \{v\}$ is a dominating set. Therefore, every vertex in D is dominated by at least one vertex in $V - D$ and $V - D$ is a dominating set. But, every vertex in D is not dominated by exactly one vertex in $V - P$. So, $V - D$ is not a perfect dominating set. Further, since $< D >$ is connected. Thus $V - D$ is not connected.
Remark 3.1: Let $G = (A, B) = K_p$ be a is a complete bipolar fuzzy graph. Let D be a minimal connected perfect dominating set of G. Then $V - D$ has a connected perfect dominating set of G.
Remark 3.2: Let $G = (A, B) = K_{n,m}$ be a complete bipartite bipolar fuzzy graph. Let D be a minimal connected perfect dominating set of G. Then $V - D$ has a connected perfect dominating set of G.
Theorem 3.3: A bipolar fuzzy graph G has connected perfect dominating set D if and only if G is a connected bipolar fuzzy graph.
Proof: Let G be a bipolar fuzzy graph with connected perfect dominating set D , since D is a perfect dominating set and $< D >$ is connected , and since every vertex v in $V - D$ is dominated by exactly one vertex in D. Thus G is a connected bipolar fuzzy

graph. Conversely, let G be a connected bipolar fuzzy graph if G is separable graph the $D = V(G) - \{u\}$ is a connected perfect dominating set of G , for every non bipolar fuzzy cut node u in $V(G)$. Hence every connected bipolar fuzzy graph has a connected perfect dominating set D.

Theorem 3.4: For any connected bipolar fuzzy graph $G = (A, B)$, $\gamma_{cp}(G) < p$.

Theorem 3.5: For any disconnected bipolar fuzzy graph $G = (A, B)$, $\gamma_{cp}(G) = 0$.

Proof: Since G is disconnected bipolar fuzzy graph, then by Theorem 3.3, G has no connected perfect dominating set. Hence $\gamma_{cp}(G) = 0$.

Theorem 3.6: For any connected bipolar fuzzy graph $G = (A, B)$

(i) $\gamma_{cp}(G) \le p - \Delta_N(G)$;

(ii)$\gamma_{cp}(G) \le p - \Delta_E(G)$.

Proof: Let $u, v \in V$ and let G be any connected bipolar fuzzy graph. We know that $\Delta_N(G)$ is the sum of the membership values of vertices excluding the maximum degree of a vertex. It is clear that $\gamma_{cp}(G) \le p - \Delta_N(G)$.

Since $\Delta_E(G) \le \Delta_N(G)$. Then $p - \Delta_N(G) \le p - \Delta_E(G)$. Hence (ii) holds.

Corollary 3.6.1: For any connected bipolar fuzzy graph $G = (A, B)$,

(i) $\gamma_{cp}(G) \le p - \delta_N(G)$;

(ii)$\gamma_{cp}(G) \le p - \delta_E(G)$.

Theorem 3.7: For any connected bipolar fuzzy graph $G = (A, B)$, $\gamma_{cp}(G) \le \alpha_0(G)$.

Proof: It is obvious.

Theorem 3.8: For any connected bipolar fuzzy graph $G = (A, B)$, $\gamma_{cp}(G) \le p - \beta_0(G)$.

Proof: Since G is connected then G has no isolated vertices, then by Theorem (3.7), $\gamma_{cp}(G) \le \alpha_0(G) = p - \beta_0(G)$.

In the following we give $\gamma_{cp}(G)$ for some standard bipolar fuzzy graphs.

Theorem 3.9: For any complete bipolar fuzzy graph $G = K_p$,

(i) $\gamma_{cp}(K_p) = min\{|v| : \forall v \in V(G)\}$;

(ii)$\gamma_{cp}(\overline{K_p}) = 0$.

Proof: Let $G = K_p$ be complete bipolar fuzzy graph. then every vertex in G dominate all the other vertices of G. Then every vertex subset D of $V(G)$ contains only one vertex is a dominating set of G, further, is a perfect and connected dominating set of G. Hence (i) holds. To prove (ii)

Since $\overline{K_P}$ is a trivial bipolar fuzzy graphs. Then $\overline{K_P}$ has no connected perfect dominating set. Hence $\gamma_{cp}(\overline{K_p}) = 0$.

Corollary 3.9.1: For any complete bipolar fuzzy graph $G = K_p$, $\gamma_{cp} = \gamma_p = \gamma_c = \gamma$.

Theorem 3.10: For any complete bipartite bipolar fuzzy graph $G = K_{n,m}$,

(i) $\gamma_{cp}(K_{n,m}) = min\{|v| : \forall v \in V_1\} + min\{|u| : \forall u \in V_2\}$;

(ii)$\gamma_{cp}(\overline{K_{n,m}}) = 0$.

Proof: Let G be a complete bipartite bipolar fuzzy graph, then $V(G) = V_1 \bigcup V_2$, $V_1 \bigcap V_2 = \phi$ and $\mu_B^P(v, u) = min\{\mu_A^P(v), \mu_A^P(u)\}$ and $\mu_B^N(v, u) = max\{\mu_A^N(v), \mu_A^N(u)\}$ for all $v \in V_1$, and $u \in V_2$.That is every vertex of V_1 dominate all vertices of V_2 and the vice versa. Furthermore every vertex in V_1 is connected with all vertices of V_2. Then we shoos $D = \{u, v : v \in V_1, u \in V_2\}$ is a connected perfect dominating set of G such that u is of minimum membership value in V_2 and v is of minimum membership value in V_1. Hence $\gamma_{cp}(K_{n,m}) = min\{|v| : \forall v \in V_1\} + min\{|u| : \forall u \in V_2\}$. Now, since $\overline{K_{n,m}}$ is disconnected. Then by Theorem 3.5, $\gamma_{cp}(\overline{K_{n,m}}) = 0$.

Corollary 3.10.1: For any complete bipartite bipolar fuzzy graph $G = K_{n,m}$,
$\gamma_{cp} = \gamma_p = \gamma_c = \gamma$.

Theorem 3.11: For any cycle bipolar fuzzy graph $G = C_n$,

(i) $\gamma_{cp}(C_n) = min\{\sum_i^{n-2} |v_i| : \forall v \in V\}$;

(ii)$\gamma_{cp}(\overline{C_n}) = 0$.

Proof: Let $G = C_n$ be a cycle bipolar fuzzy graph, then the connected perfect dominating set of G contain exactly $n-2$ vertices of G started from v_1 or others. Then the connected perfect domination number equal to the minimum cardinality of all vertex subsets D_i of V which contain $n-2$ vertices whether we started from v_1 or v_2 or v_3,...and so on. Hence $\gamma_{cp}(C_n) = min\{\sum_{i}^{n-2} |v_i| : \forall v \in V\}$. Now, if $G = C_n$, then $\overline{C_n}$ is disconnected. Then by Theorem 3.5., $\gamma_{cp}(\overline{C_n}) = 0$.

Corollary 3.11.1: For any cycle bipolar fuzzy graph $G = C_n$,
$\gamma_{cp} = \gamma_c$.

Theorem 3.12: For any connected bipolar fuzzy graph $G = (A, B)$, $\gamma_{cp}(G) + \gamma_{cp}(\overline{G}) \leq p$.

Theorem 3.13: Let $G : (A, B)$ be any connected bipolar fuzzy graph and $H : (A', B')$ be any maximum spanning tree of G. Then every connected perfect dominating set of H is a connected perfect dominating set of a bipolar fuzzy graph G and $\gamma_{cp}(G) \leq \gamma_{cp}(H)$.

Proof: Let D be a connected perfect dominating set of a bipolar H, since H is a maximum spanning tree of G .we have $\mu_A^p = \mu_{A'}^P$ and $\mu_A^N = \mu_{A'}^N$. Thus the vertices in $V - D$ is dominated by exactly one vertex in D and the induced bipolar fuzzy subgraph $< D >$ is connected. Hence D is connected perfect dominating set of a bipolar fuzzy graph G. Therefor, $\gamma_{cp}(G) \leq \gamma_{cp}(H)$.

Theorem 3.14: For a non-trivial bipolar fuzzy tree $T = (A, B)$ for $n \geq 3$, the set of all cut vertices in T is a connected perfect dominating set of bipolar fuzzy Tree T.

Proof: If D is the set of all bipolar fuzzy cut vertices of a bipolar fuzzy tree $T = (A, B)$, then D is a perfect dominating set of T. Claim that the induced bipolar fuzzy subgraph $< D >$ is connected. Internal vertices are the bipolar fuzzy cut vertices of T. Thus the induced bipolar fuzzy subgraph has the internal vertices is connected. Therefor $< D >$ is connected. Hence D is connected perfect dominating set of a bipolar fuzzy tree T.

Theorem 3.15: For a non-trivial bipolar fuzzy tree $G = (A, B)$ each vertex of every connected perfect dominating set D is contained in the unique maximum spanning tree of a bipolar fuzzy graph G.

Proof: Since G is a bipolar fuzzy tree, G has a unique maximum spanning tree which contains all the vertices of G. Therefor, unique maximum spanning tree contains all the vertices of every connected perfect dominating set of G.

Theorem 3.16: For a non-trivial bipolar fuzzy tree $G = (A, B)$ each vertex of a connected perfect dominating set D is incident on a fuzzy bride of G.

Proof: Let D be a connected perfect dominating set of a bipolar fuzzy tree G. Let $u \in D$. Since D is a connected perfect dominating set, a vertex $v \in V - D$ is dominated by exactly on vertex of D such that (u, v) is strong edge. Then (u, v) is an edge of the unique maximum spanning tree F of a bipolar fuzzy graph G. Hence (u, v) is a bipolar fuzzy bridge of G. This is true for every vertex of a connected perfect dominating set D of a bipolar fuzzy graph G. Thus the theorem is true.

Theorem 3.17: For any strong bipolar fuzzy tree $G = (A, B)$ such that G^* is a tree with more than 3 vertices then $\gamma_{cp}(G) = p - p^*$, where P is order the bipolar fuzzy graph and P^* is the number of pendant vertices of G.

Proof: Let $G = (A, B)$ be strong star bipolar fuzzy tree, and let S the set of all pendant vertices of G, since G is strong, then $V - S$ is perfect dominating set of G and is connected. Therefore, $V - S$ is a connected perfect dominating set. Hence, $\gamma_{cp}(G) = |V - S| = p - p^*$.

Theorem 3.18: Let $G = (A, B) = W_{n+1}$ be a strong wheel bipolar fuzzy graph, then $\gamma_{cp}(G) = |u|$, u is a root vertex.

Proof: Let $G = (A, B) = W_{n+1}$ be a strong wheel bipolar fuzzy graph and u is a root of G, then u dominates all the other vertices of G, therefor $D = \{u\}$ is connected perfect dominating set of G. Hence, $\gamma_{cp}(G) = |u|$, u is a root vertex.

Theorem 3.19: Let $G = (A, B) = S_{n+1}$ be a strong Star bipolar fuzzy graph, then

$\gamma_{cp}(G) = |u|$, u is a root vertex.

Proof: Let $G = (A, B) = S_{n+1}$ be a strong star bipolar fuzzy graph and u is a root vertex of G, then u dominates all the other vertices of G, therefor $D = \{u\}$ is connected perfect dominating set of G. Hence, $\gamma_{cp}(G) = |u|$, u is a root vertex.

Theorem 3.20: If D is a minimal connected perfect dominating set of a bipolar fuzzy graph G then D is a perfect dominating set and $< D >$ is a path.

Proof: It is clear that every connected perfect dominating set of a bipolar fuzzy graph G is perfect dominating set of a bipolar fuzzy graph G. if D be a minimal connected perfect dominating set of G with $n \geq 3$ then D is a path or cycle. Suppose that $< D >$ is a cycle say $C = u_1u_2u_2.....u_nu_1$. Let v_i be the unique vertex in $V - D$ such that v_i is dominates u_i . Clearly v_i is not dominates u_j for $i \neq j$ and hence v_i dominate to two vertices in $V - D$ say, u_j and u_k. Now $D - \{u_j, u_k\} \bigcup \{v_i\}$ is dominating set with cardinality $\gamma_c(G) - |u|$ which is a contradiction. Hence D is a path.

Theorem 3.21: Every connected perfect dominating set of a bipolar fuzzy graph $G = (A, B)$ is a dominating set of G.

Proof: Let G be a bipolar fuzzy graph and D is connected perfect dominating set of G, then every vertex $v \in V - D$ is dominated by exactly one vertex $u \in D$ and $< D >$ is connected bipolar fuzzy subgraph. Then D is dominating set of G.

Theorem 3.22: Every connected perfect dominating set of a bipolar fuzzy graph $G = (A, B)$ is a connected dominating set of G.

Proof: Let G be a bipolar fuzzy graph and D is connected perfect dominating set of G, then every vertex $v \in V - D$ is dominated by exactly one vertex $u \in D$ and $< D >$ is connected bipolar fuzzy subgraph. Then D is a connected dominating set of G.

Theorem 3.23: Let $G = (A, B)$ be connected bipolar fuzzy graph, then
(i) $\gamma_{cp} \geq \gamma(G)$;
(ii) $\gamma_{cp} \geq \gamma_p(G)$;
(iii) $\gamma_{cp} \geq \gamma_c(G)$.

Proof: Let $G = (A, B)$ be a connected bipolar fuzzy graph and D is connected perfect dominating set of G. Then By Theorem 3.22 D is dominating set of G. Hence $\gamma_{cp} \geq \gamma(G)$. Thus (i) holds. Now By Theorem 3.1 D is perfect dominating set of G. Hence $\gamma_{cp} \geq \gamma_p(G)$. Thus (ii) holds. Finally, By Theorem 3.22 D is connected dominating set of G. Hence $\gamma_{cp} \geq \gamma_c(G)$. Thus (iii) holds.

Theorem 3.24: Let $G = (A, B)$ be connected bipolar fuzzy graph, then

$$\gamma_{cp}(G) \geq \frac{p}{\Delta_N + 1}.$$

Proof: Let D be a connected perfect dominating set of a connected bipolar fuzzy graph G with cardinality $|D| = \gamma_{cp}(G)$ and since every vertex of $V - D$ is dominated by exactly one vertex of D. We have $|V - D| \leq \gamma_{cp}(G).\Delta_N(G)$, then $p - \gamma_{cp}(G) \leq \gamma_{cp}(G).\Delta_N(G)$. Thus $p \leq \gamma_{cp}(G).(\Delta_N(G) + 1)$. Hence $\gamma_{cp}(G) \geq \frac{p}{\Delta_N + 1}$.

Theorem 3.25: Let $G = (A, B)$ be connected bipolar fuzzy graph, then

$$\gamma_{cp}(G) \leq \frac{p.\Delta_N}{\Delta_N + 1}.$$

Proof: Let D be a connected perfect dominating set of a connected bipolar fuzzy graph G with cardinality $|D| = \gamma_{cp}(G)$ and since every vertex of $V - D$ is dominated by exactly one vertex of D. We have $|V - D|.\Delta_N(G) \geq \gamma_{cp}(G).\Delta_N(G)$, then $(p - \gamma_{cp}(G)).\Delta_N(G) \geq \gamma_{cp}(G).\Delta_N(G)$. Thus $p.\Delta_N(G) \geq \gamma_{cp}(G).(\Delta_N(G) + 1)$. Hence $\gamma_{cp}(G) \leq \frac{p.\Delta_N(G)}{\Delta_N + 1}$.

Theorem 3.26: Let $G = (A, B)$ be a connected bipolar fuzzy graph. Then a connected perfect dominating set D is a minimal connected perfect dominating set of G if and only if D is a fuzzy irredundant set.

Proof: Let D be a minimal connected perfect dominating set of a connected bipolar fuzzy graph G. Therefore, for every vertex $v \in D$, there exists a vertex $u \in V - \{D - \{v\}\}$ which is not dominated by $B - \{v\}$. That is every vertex $v \in D$ has at least one private neighbor. In other words $P_N[v, D] \neq \phi$. Hence D is fuzzy irredundant set.

Conversely, if D is a fuzzy irredundant set. Suppose that D is not minimal connected perfect dominating set of G. Then there exists a vertex $v \in D$, for which $D - \{v\}$ is connected perfect dominating set. But D is fuzzy irredundant set. Therefor, $P_N[v, D] \neq \phi$. Let $u \in P_N[v, D]$, then by definition u is not strong neighbor to any vertex in $D - \{v\}$. That is D is not dominating set, which is a contradiction. Therefor, D is minimal connected perfect dominating set of a connected bipolar fuzzy graph G.

Theorem 3.27: If D is a connected perfect dominating set of a Petersen strong bipolar fuzzy graph $G = (A, B)$, then $V - D$ is connected perfect dominating set of G.

Proof: Let D is a connected perfect dominating set of a Petersen strong bipolar fuzzy graph $G = (A, B)$. In $V - D$ is also satisfies the definition of connected perfect dominating set of a strong bipolar fuzzy graph G. That is, for every vertex $v \in V - D$, v dominated by exactly one vertex of D and also the bipolar fuzzy subgraph $< D >$ induced by D is connected. Therefore, $V - D$ is connected perfect dominating set of a strong Petersen bipolar fuzzy graph G.

Theorem 3.28: For any complete bipolar fuzzy graph $G = K_p$;Then $\gamma_{cp}(K_p) \geqslant O^p(K_p) - \Delta^p(K_p) \forall v \in V(G)\}$.

Proof: Let v by any vertex without minimal in complete bipolar fuzzy graph. Assume that $\mid N(v) \mid = \Delta^p(K_p)$ and$\mid V \mid = O^P(K_p)$. Then $V - N(v)$ is connected perfect dominating set in $G = K_p$ $(i.e)$ v is connected perfect dominating set in $G = K_p$. So $\gamma_{cp}(K_p) \geqslant \mid V - N(v) \mid \geqslant \mid V \mid - \mid N(v) \mid = O^p(K_p) - \Delta^p(K_p) \Longrightarrow \gamma_{cp}(K_p) \geqslant O^p(K_p) - \Delta^p(K_p)$.

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